Newton's Shell Theorems

Consider a uniform spherical shell of mass density σ and radius R (for a total shell mass of $M = 4\pi R^2\sigma$), centered at the origin O. A mass m is located at point P a distance r from O. Now, choose a band on the sphere such that it is perpendicular to the line OP; that is, if OP is the sphere's 'axis', the band follows lines of latitude. Pick a particle Q on this band, with mass dm_s. The force between masses P and Q is given by

$$dF = Gm\frac{dm_s}{{r_0}^2}\cos\theta_0 \tag{1}$$

where r_0 is the distance between P and Q, and θ_0 is the angle between OP and QP.

Let θ be the angle between the lines OP and OQ. To find the total force on P from the shell, (1) must be integrated over $0 \leq \theta \leq \pi$. From the law of cosines:

$$\begin{aligned} {r_0}^2 &= r^2 + R^2 - 2rR\cos\theta && (2) \\ R^2 &= r^2 + {r_0}^2 + 2rr_0\cos\theta_0 && (3) \end{aligned}$$

Solving for $\cos\theta_0$:

$$\cos\theta_0 = \frac{r - R\cos\theta}{r_0} \tag{4}$$

Substituting in (2):

$$\cos\theta_0 = \frac{r - R\cos\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} \tag{5}$$

The mass of the band is given by

$$dM = \sigma(2\pi R\sin\theta)(R\,d\theta) \tag{6}$$

Since the band is symmetric about OP the net force perpendicular to the line is zero. Parallel to it, the component of the force due to dM is

$$\begin{aligned} dF &= Gm\frac{dM}{{r_0}^2}\cos\theta_0 \\ &= Gm\frac{\sigma(2\pi R\sin\theta)(R\,d\theta)}{r^2+R^2-2rR\cos\theta}\frac{r-R\cos\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} \end{aligned} \tag{7}$$

Integrating:

$$F = Gm\int_0^\pi \frac{\sigma(2\pi R\sin\theta)(R\,d\theta)}{r^2+R^2-2rR\cos\theta}\frac{r-R\cos\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} \tag{8}$$

To simplify matters, let

$$\rho = \frac{R}{r} \tag{9}$$

$$a = \frac{1+\rho^2}{2} \tag{10}$$

$$x = \rho\cos\theta \tag{11}$$

$$dx = -\rho\sin\theta\,d\theta \tag{12}$$

Substituting and rearranging, we get

$$\begin{aligned} F &= \frac{GMm}{r^2}\frac{1}{2^{\frac{5}{2}}\rho}\int_{-\rho}^{\rho}\frac{1-x}{(a-x)^{\frac{3}{2}}}\,dx \\ &= \frac{GMm}{r^2}\frac{1}{2^{\frac{3}{2}}\rho}\left[\frac{1-2a+x}{\sqrt{a-x}}\right]_{-\rho}^{\rho} \\ &= \frac{GMm}{r^2}\frac{1}{2^{\frac{3}{2}}\rho}\left(\frac{1-2a+\rho}{\sqrt{a-\rho}}-\frac{1-2a-\rho}{\sqrt{a+\rho}}\right) \end{aligned} \tag{13}$$

From (9) and (10) we see that

$$1-2a+\rho = -\rho(\rho-1) \tag{14}$$

$$1-2a-\rho = -\rho(\rho+1) \tag{15}$$

$$a+\rho = \tfrac{1}{2}(\rho+1)^2 \tag{16}$$

$$a-\rho = \tfrac{1}{2}(\rho-1)^2 \tag{17}$$

Since (16) and (17) are necessarily positive,

$$\sqrt{a+\rho} = \sqrt{\tfrac{1}{2}}(\rho+1) \tag{18}$$

$$\begin{aligned}\sqrt{a-\rho} &= \sqrt{\tfrac{1}{2}}\begin{cases}\rho-1, & \rho>1\\ 1-\rho, & \rho<1\end{cases}\\ &= \pm\sqrt{\tfrac{1}{2}}(\rho-1)\end{aligned} \tag{19}$$

where the sign chosen is the same as that of $\rho - 1$.

Substituting into (13):

$$\begin{aligned}F &= \frac{GMm}{r^2}\frac{1}{2\rho}\left(\frac{-\rho(\rho-1)}{\pm(\rho-1)} - \frac{-\rho(\rho+1)}{\rho+1}\right)\\ &= \frac{GMm}{r^2}\left(\frac{1\mp 1}{2}\right)\end{aligned} \tag{20}$$

If $r > R$, $\rho < 1$ and the sign chosen in (19) is negative; the rightmost term becomes one, and F becomes the familiar expression for the force exerted by a point mass. If $r < R$ (that is, P is inside the shell), then $\rho > 1$ and the rightmost term becomes zero. Thus:

$$F = \begin{cases}\frac{GMm}{r^2}, & r>R\\ 0, & r<R\end{cases} \tag{21}$$